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AN EXTENSION OF THE MARKOWITZ PORTFOLIO SELECTION MODEL
TO INCLUDE VARIABLE TRANSACTIONS' COSTS, SHORT SALES,
LEVERAGE POLICIES AND TAXES

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Introduction

The approach to portfolio selection upon which most of the current academic work in this area is based was developed by H. M. Markowtiz and presented in a 1952 paper.[1] Since that time many extensions to Markowitz's basic approach have been suggested by various authors attempting to explain the asset-holding behavior of individuals or develop normative rules for asset choice.

In much of this work a standard set of assumptions about the securities markets continually reappears. These assumptions relate to the costs involved in revising an existing portfolio to obtain another which is more desirable in terms of revised expectations about future security prices. The assumptions relate to two types of portfolio transactions' costs; the brokerage fees involved in exchanging portfolio assets and price effects resulting from asset illiquidities.

Current portfolio selection models generally ignore the brokerage fees involved in revising an existing portfolio. The result of this assumption is that frequent portfolio revisions may occur which are not justified relative to the resulting brokerage fees. Small changes in expectations about a particular security can result in transactions which would not occur if the broker's fees for purchasing or selling that asset were considered.

The second cost relates to the liquidity of portfolio assets. It is usually assumed that assets are perfectly liquid, that is, convertible without delay into currency at full market value, in any quantity. This as-

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[1] Harry M. Markowitz, "Portfolio Selection, <u>The Journal of Finance</u>, March 1952.

sumption is challenged by many institutional investors. Depending on the nature of the security involved, institutional investors contend that substantial unfavorable price spreads can result in attempts to buy or sell large quantities of stock. If volume related price effects exist, then portfolio selection models which neglect these costs can produce portfolio turnover rates which are non-optimal in terms of the price-spread transactions costs involved. This consideration is of particular importance to large institutional investors.[1]

In addition to these assumptions regarding portfolio transactions costs, a restricted set of investment alternatives is usually considered. Excluded are short sales and liability holdings, including secured margin loans and other types of unsecured debt. Substantial use of these techniques by individuals and financial institutions[2] exists in the capital markets.

When the set of investment alternatives is expanded to include short sales and liabilities, the resulting set of efficient portfolios will generally dominate the set created in their absence. Thus, for a given risk level, portfolios selected under the expanded set of investment alternatives will have expected returns which are equal to or greater than the portfolios selected under the usual restrictions.

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[1]While empirical evidence indicates the existence of price effects for large transactions, they are generally smaller than the effects hypothesized by many institutional investors. The question of impact on the market of large blacks of stock is currently receiving the attention of a number of researchers and institutions, including the Securities and Exchange Commission.

[2]For example, a set of investment companies, usually designated as hedge funds, make particular use of these procedures.

Finally, there is the question of taxes on portfolio capital gains and dividend income. When capital gains taxes are considered, transactions produced by a model which ignores taxes may no longer be optimal. The effect of differential tax rates on capital gains and dividend income is a factor which is relevant when portfolios are selected or revised.

The purpose of this paper is to consider a number of these generally neglected issues. The Markowitz model will be extended to include the investor's expectations regarding the two components of portfolio transactions costs--brokerage charges and price effects associated with large volume transactions. The model will include short sale and liability alternatives, as well as a treatment of the tax problem.

Investor Preferences and Subjective Beliefs

The following assumptions about investor preferences and subjective prior beliefs regarding security returns are required.

A 1. The investor attempts to maximize his expected utility of terminal wealth, in the von Neumann-Morgenstern sense. Here terminal wealth is considered to be identical to the market value of the investor's portfolio at the end of his planning horizon.

A 2. The investors planning horizon consists of a single period. The investment strategy involves selection of an optimal portfolio at the beginning of the period which will be held unchanged to the terminal date.

A 3. The investor is assumed to be risk averse. The investor's marginal utility of wealth is assumed to be everywhere non-negative and a decreasing function of wealth.

In addition, one of the following assumptions is made.

B 1. The investor's subjective prior joint distribution of one-period security returns is multivariate normal. It then follows that the distributions of portfolio returns will be normal as well.[1]

B 2. The subjective distribution of one-period security returns are such that the returns on feasible portfolios will be normally distributed.[2]

B 3. The investor's utility function can be well approximated by a quadratic function in the range of portfolio returns.[3]

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[1]One-period security return, $\tilde{R}_j$, is a linear transformation of terminal security value $\tilde{M}_j$, where $\tilde{M}_j = \tilde{P}_j + \tilde{D}_j$ and

$\tilde{P}_j$ = Terminal market price of security j.
$\tilde{D}_j$ = dividends paid during the period.

Thus, the investors one-period expected utility maximization problem can be defined in terms of one-period portfolio return, $\tilde{R}_p$, as well as in terms of terminal portfolio value $\tilde{M}_p$. Similarly, if the security returns, $\tilde{R}_j$, j=1, . . ., N are jointly normally distributed, the terminal security values, $\tilde{M}_j$, will be as well.

[2]This requirement is potentially considerably less restrictive than that implied by assumption B 1. It is probably most applicable in the case of large institutional investors, who hold many securities in their portfolios, (e.g., a hundred or more) none of which contributes in a major way to the distribution of total portfolio return. This condition relies on a generalization of the central limit theorem to random variables which are not identically or independently distributed. In the case of independently (but not identically) distributed random variables, we can rely on Lindeberg's generalization of the central limit theorem (See W. Feller, An Introduction to Probability Theory and Its Application, Vol II, pp. 256-257). For the more realistic case of non-independence the limit theorems become more complex and, as a practical matter, the question of portfolio normality is probably best investigated via simulation.

[3]Along with this assumption, it will also be necessary to assume the existence of means and standard deviations for the investor's prior distributions of one-period security returns.

Conditions B 1 and B 2 place restrictions on the investor's subjective probability distributions. Condition B 3 places parametric restrictions on his utility of return function. Tobin[1] has shown that when one of these assumptions is valid, the investors preference for portfolios can be determined solely on the basis of the one-period means and standard deviations of return. The optimal portfolio will be a member of the mean-standard deviation efficient set, where an efficient portfolio must satisfy the following criteria. (1) if any other portfolio provides a lower standard deviation of one period return, it must also have a lower expected return; and (2) if any other portfolio has greater expected return, it must also have greater standard deviation of return

The following are the major notational symbols used throughout the paper.

N = number of securities in the universe considered.

$\tilde{P}_j$ = the price of security j at the end of the planning horizon.

$\tilde{D}_j$ = the dividends paid on security j during the time horizon.

$\tilde{M}_j$ = the terminal market value of security j

$= \tilde{P}_j + D_j$

$\hat{M}_j$ = the mean of the investor's prior distribution for $\tilde{M}_j$

$= \hat{P}_j + \hat{D}_j$

σ_{jj} = the variance of the investor's distribution for $\tilde{M}_j$,

$= E(\tilde{M}_j - \hat{M}_j)^2$.

$\sigma_{jj'}$ = the covariance between $\tilde{M}_j$ and $\tilde{M}_{j'}$,

$= E(\tilde{M}_{j'} - \hat{M}_{j'})(\tilde{M}_j - \hat{M}_j)$.

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[1] James Tobin, "Liquidity Preferences as Behavior Toward Risk," <u>Review of Economic Studies</u>, (Feb 1958) pp. 65-86.

X_j = the number of shares of security j held during the investment period.

$X_j(0)$ = the number of shares of security held prior to the investment period (before the portfolio is revised).

$P_j(0)$ = the price of security j at the beginning of the investment period.

For compactness of notation, the following vector quantities are defines.

$\underline{X}'$ = the revised portfolio vector,

$= (X_1, \ldots, X_N).$

$\underline{X}'(0)$ = the initial portfolio vector,

$= (X_1(0), \ldots, X_N(0)).$

$\underline{\tilde{M}}$ = the vector of terminal security values,

$= (\tilde{M}_1, \ldots, \tilde{M}_N).$

$\underline{P}(0)$ = the vector of initial security prices,

$= (P_1(0), \ldots, P_N(0)).$

$\underline{\Sigma}$ = the covariance matrix of security terminal values,

$= \| \sigma_{jj'} \|$

$$= \| E(\tilde{M}_j - \hat{M}_j)(\tilde{M}_{j'} - \hat{M}_{j'}) \| \qquad \begin{matrix} j=1, \ldots, N \\ j'=1, \ldots, N \end{matrix}$$

Thus the investor's estimate of the portfolio market value at the end of the investment period is given by

$$\hat{M}_p = \sum_{j=1}^{N} X_j \hat{M}_j = \underline{X}^1 \underline{\hat{M}}$$

$$= \sum_{j=1}^{N} X_j (\hat{P}_j + \hat{D}_j) = \underline{X}^1 (\underline{\hat{P}} + \underline{\hat{D}})$$

The variance of the investor's prior distribution of portfolio return is

$$V_p = \sum_{j=1}^{N} \sum_{j'=1}^{N} X_j X_{j'} \sigma_{jj'} ,$$

$$= \underline{X}^1 \underline{\Sigma} \, \underline{X}$$

The efficient pairs $(\hat{M}_p, V_p)$ and the corresponding portfolio vectors $\underline{X}$ which yield them are determined by solving the problem

$$\text{Max } Z = \Theta \, \underline{X}^1 \underline{M} - \underline{X}^1 \underline{\Sigma} \, \underline{X}$$

$$\text{for all} \quad \Theta \geq 0$$

subject to the set of resource, policy and legal restrictions which are relevant for the investor. In the model developed in this paper, most of the constraints are linear functions of the decision variables, $X_1, \ldots, X_N$, and thus can be summarized as

$$\underline{AX} \leq \underline{B}$$

where $\underline{A}$ = a matrix of resource utilization coefficients

$\underline{B}$ = a vector of resource limitation or other activity upper bounds.

We now proceed to develop the form of the vectors A and B via the consideration of transactions costs, taxes and various types of investment and financing alternatives.

Transactions Costs

As previously discussed, security transactions costs are considered as comprising of two parts, an asset exhcnage or brokerage fee and a liquidity or marketability cost.

(a) Brokerage Fees

For ease of exposition of the total model, only the case of proportional brokerage fees will be considered in the main text. A formulation of the volume discount case is presented in appendix A.

Let c_j = the fraction of the pershare auction market price which must be paid in brokerage fees to transact one share of security j.

Therefore, the cost of purchasing x_j^+ shares of security j is given $c_j\ x_j^+$.

(b) Marketability Costs

The difficulty in purchasing or selling a given quantity of stock in a specified period is generally considered to be related to the liquidity of the auction market, which, for a specific security can be measured in terms of the "normal" trading volume of the stock. A particular transaction which represents 10-20% of the average trading volume in a given period can, in most cases, be more easily transacted than a trade which represents many times the normal auc-

tion market volume. The additional expense results from the costs of informing additional purchasers or sellers about the current unusual opportunities that exist and offering them inducements to rebalance their portfolios, which can consist of favorable price spreads and/or payment of any brokerage fees resulting from the trade. In addition, in relation to purchases of large blocks of a stock, some additional incentive may be required to induce individuals with capital gains liabilities to provide their shares.

In this model, for each security, we use the expected normal trading volume as a metric with which to relate expected marketability costs to volume of shares traded.[1] Note that since an expected transactions' costs curve is being defined for each security, the investor can incorporate any expectations he may hold regarding the special ease or difficulty of trading large volumes of a particular stock.

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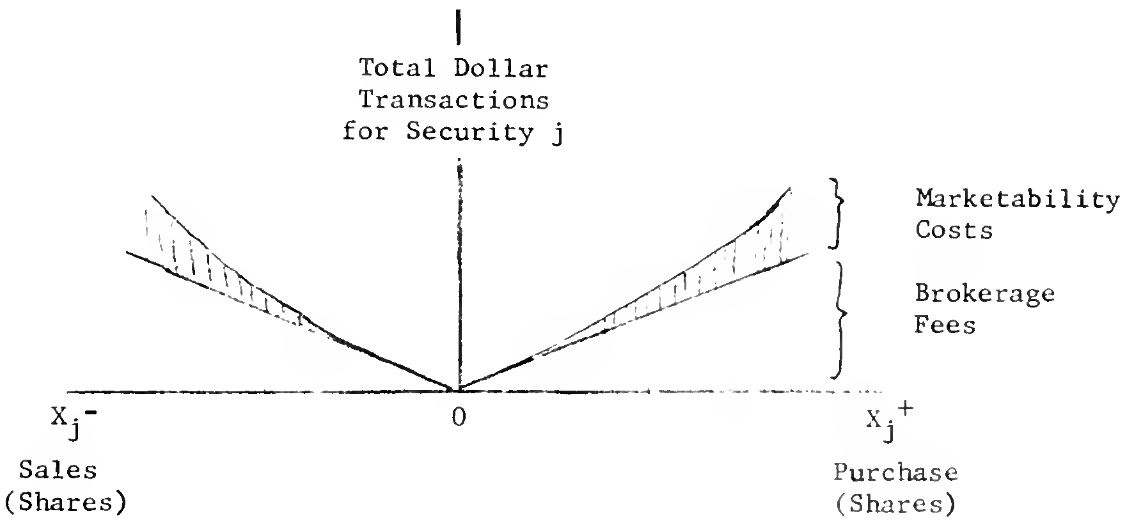
[1]Additional measures of the relative size of a transaction could be used instead of the proportion of "normal" trading volume. An example is the percentage of stock outstanding represented by the trade.

The type of total transactions' cost curve used in the model is illustrated in Figure 1. The shaded area represents the investor's expectation of the costs that will be necessary to purchase or sell a given volume of shares of security j, in addition to brokerage fees.

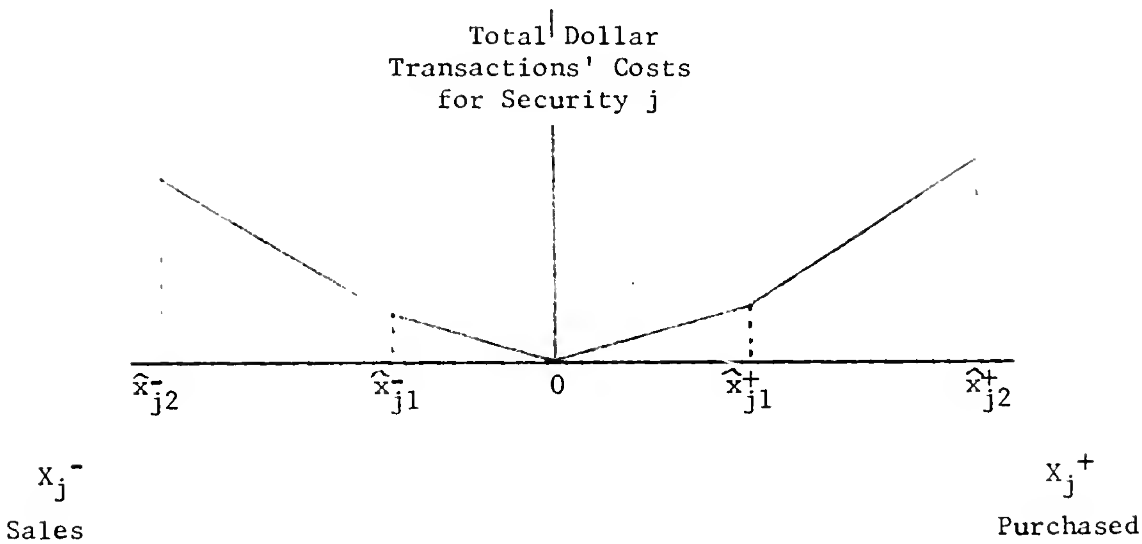
Figure 1

TOTAL TRANSACTIONS' COSTS CURVE



In Figure 2, the above curve has been approximated by a piece-wise linear representation. The change points for the marginal transactions' costs rates (i.e., the slopes of the linear segments) occur when purchases or sales of security j amount to specified percentages of the expected normal trading volume for that security.

Figure 2
PIECEWISE LINEAR APPROXIMATION
TO THE TRANSACTIONS' COSTS CURVE



Let c^+_{ji} = the percentage of the current auction market price, $P_j(0)$, which must be paid for transactions in the i^{th} linear segment of the total transactions' costs curve for security j ($i=1, \ldots, m^+$)

r^+_{ji} = the dollar transactions costs per share for purchases in the i^{th} linear segment

= $c^+_{ji}P_j(0)$

$\hat{x}^+_{ji}$ = the number of shares of security j which corresponds to a specified fraction S^+_i of the normal trading volume of security j. $\hat{x}^+_{ji}$ defines the upper limit of the i^{th} purchase segment of the cost curve.

x^+_{ji} = the number of shares of security j purchased in the i^{th} linear segment of the cost curve.

x^+_j = the total number of shares of security j purchased

$$= \sum_{i=1}^{m^+} x^+_{ji}$$

Similar quantities can be defined for the sales segments of the transactions' cost curve.

We can now define the number of shares of security j traded in terms of purchases or sales in the linear segments of the cost curve.

The number of shares of security j traded

$$= X_j - X_j(0)$$
$$= x_j^+ - x_j^-$$

$$= \sum_{i=1}^{m+} x_{ji}^+ - \sum_{i=1}^{m-} x_{ji}^- \qquad (1)$$

The transactions' costs incurred

$$= \sum_{j=1}^{m+} r_{ji}^+ x_{ji}^+ + \sum_{j=1}^{m-} r_{ji}^- x_{ji}^- \qquad (2)$$

The transactions costs will be included in the budget equation, (described below) reducing the amount of resources available for reinvestment in a revised portfolio.

Additionally, we require that each of the transaction's variables x_{ji}^+ and x_{ji}^- be upper bounded

$$x_{ji}^+ \leq \hat{x}_{ji}^+ \qquad i=1, \ldots, m^+ \qquad (3)$$
$$x_{ji}^- \leq \hat{x}_{ji}^- \qquad i=1, \ldots, m^-$$

Because of the convexity of the transactions' cost curve, we need not be concerned about the possibility that $x_{j\ i+1}^+ \geq 0$ while $x_{ji}^+ \leq \hat{x}_{ji}^+$. This condition will not arise because higher segments of the curve are more costly in terms of transactions' costs.

Taxes

The investor is assumed to be interested in the terminal market value of his portfolio, net of income taxes on dividend income received during the period and capital gains on portfolio appreciation. Also, when portfolio revisions are made at the beginning of the investment period, capital gains tax liabilities (or credits) will result from the realization of gains (or losses) on the securities traded.[1]

Define $P_j(A)$ = the average purchase price of the investor's initial holding of security j

Let T_c = the investor's margin tax rate on capital gains

T_I = the investor's marginal tax rate on income.

When the initial portfolio, $\underline{X}(0)$, is revised, the cash flow resulting from capital gains or losses on securities held is given by

$$T_c \sum_{j=1}^{N} x_j^-[P_j(0) - P_j(A)]$$

where x_j^- is the number of shares of security j which are sold. This term will be included in the budget equation discussed below.

The market value of the terminal portfolio, net of tax liabilities is given by

$$\tilde{M}_P^T = \sum_{j=1}^{N} X_j\hat{P}_j + \left[\sum_{j=1}^{N} X_j\tilde{D}_j\right](1-T_I) - T_c\left[\sum_{j=1}^{N} X_j(\tilde{P}_j-P_j(0)) + \sum_{j=1}^{N} (X_j(0)-X_j^-)(P_j(0)-P_j(A))\right]$$

[1]For simplicity all capital gains are assumed to be long term. Extension of the model to include short term gains is straightforward.

The first term is the market value of the terminal portfolio. The second term is the net of taxes dividend income received during the investment period. The first part of the third term represents capital gains taxes due on security appreciation during the investment period. The second part of the term represents capital gains taxes on unrealized appreciation in the starting portfolio.

Recalling that

$$X_j - X_j(0) = x_j^+ - x_j^- ,$$

the above expression can be simplified to give

$$\tilde{M}_P^T = (1-T_c)\left[\sum_{j=1}^{N} X_j\tilde{P}_j\right] + (1-T_I)\left[\sum_{j=1}^{N} X_j\tilde{D}_j\right]$$
$$+ T_c\left[\sum_{j=1}^{N} (X_j - x_j^+)P_j(A) + \sum_{j=1}^{N} x_j^+P_j(0)\right]$$

$$= \underline{X}^1 \, [(1-T_c)\underline{\tilde{P}} + (1-T_I)\underline{\tilde{D}}] + T_c \, [(\underline{X}-\underline{X}^+)'\underline{P}(A) + \underline{X}^+\underline{P}(0)] \qquad (4)$$

Short Sales

The allowance for short sales can be incorporated by defining an additional set of N securities which are simply short positions in the original securities.

Define X_{N+j}, $j=1, \ldots, N$ as the number of shares of security j held short during the investment period.

The return on a share of security j, $\tilde{R}_j$, and the return on a share of security j sold short, $\tilde{R}_{N+j}$, have the following relationships

Expected Return	Variance of Return
$E(\tilde{R}_j) = -E(\tilde{R}_{N+j})$	$\sigma^2(\tilde{R}_j) = \sigma^2(\tilde{R}_{N+j})$

Pair Wise Correlations Between Returns

$$\rho(\tilde{R}_j, \tilde{R}_{N+j}) = -1$$

When the investor sells securities short, the proceeds of the short sale are retained by the broker affecting the sale until the short position is closed out. The proceeds must be adjusted as market prices change so that the value is equal to the market value of the securities sold short. In addition to the proceeds, the investor must provide collateral to the broker equal to the market value of the securities sold short.[1] For ease of future exposition, any short positions are assumed to be collateralized at the beginning of the investment period with cash.[2] The investor earns no interest on the proceeds held by the broker (the credit balance in his short account) but earns interest at the rate on broker's loans, rm, on collateral held by the broker (the credit balance in his margin account.)

Define C(0) as the amount of proceeds and collateral balances held by broker before the portfolio revision,[3] and C(1) as the required balance after

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[1]Short sales on margin are considered in the next section.

[2]The model can be generalized without substantial difficulty to allow for collateral in the form of unencumbered securities.

[3]Since the model is a discrete and not continuous time period model, C(0) will equal the proceeds and collateral balance existing after the previous portfolio revision, i.e., one investment period ago. Given security prices have adjusted during the period, the existing short positions may thus be under or over collateralized prior to the current revision. Therefore, one of the functions of the current revision is to adjust the proceeds and collateral deposit balance on existing, as well as for new short positions. If prices have fallen during the period, funds can be withdrawn from collateral balances for investment purposes, and vice versa.

the portfolio is revised

$$C(1) = 2.0 \sum_{j=N+1}^{2N} X_j P_j(0)$$

In the budget equation a term equal to C(1) - C(0) must be included to allow for the absorption or generation of portfolio cash due to changes in the proceeds and collateral requirements when the portfolio is revised.

The investor's balance sheet (see Exhibit 1) now includes liabilities equal to the amount of his short position.

Exhibit 1

BALANCE SHEET AT BEGINNING
INVESTMENT PERIOD
(After Portfolio Revision)

Assets	Liabilities
Portfolio Market Value	Short Position
$\sum_{j=1}^{N} X_j P_j(0)$	$\sum_{j=N+1}^{2N} X_j P_j(0)$
Proceeds and Collateral Deposits	Net Worth
C(1)	NW(1)

Portfolio Debt -- Secured Loans

Portfolio Leverage can be obtained by making portfolio purchases and short sales on margin. With margin purchases, a specified portion of the purcahse price is advanced (the initial margin) and the remainder is borrowed from the broker. The securities purchased become collateral for the loan and remain with the broker. The broker is compensated via an interest charge on the amount of the loan (the account net debit balance). When securities are purchased, the initial margin advanced must be equal or greater than the minimum initial margin requirement specified by the Federal Reserve Board.[1]

For long positions, the equity status of the margin account is given by

$$\text{Margin} = \frac{\text{Value of Collateral} - \text{Net Debit Balance}}{\text{Value of Collateral}}$$

$$= \frac{\text{Margin Account Equity Balance}}{\text{Value of Collateral}}$$

In the period following purchase, declines in market value can reduce the account equity status until the maintenance margin level is reached. Maintenance margin levels are determined by the national securities exchanges and represent minimum acceptable account equity ratios. Beyond this point the broker must issue a "margin call" requiring that the borrower increase the equity status of his account.

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[1]Federal Reserve Regulations T, U and G.

Short positions can also be carried on margin. In this case only a specified fraction of the normal collateral need by deposited with the broker. The collateral deposited is credited to the investor's margin account and generates interest credits which can be used to offset interest charges on funds borrowed to carry long positions on margin.[1]

For short positions, the account margin status is given by

$$\text{Margin} = \frac{\begin{matrix}\text{Proceeds} \\ \text{of Sale}\end{matrix} + \text{Collateral} - \begin{matrix}\text{Market Value} \\ \text{Short Positions}\end{matrix}}{\text{Market Value Short Positions}}$$

Now, if the account is properly maintained the proceeds deposited with the broker should always be equal to the market value of securities short.[2] In this event, the margin status of the account is given by

$$\text{Margin} = \frac{\text{Value of Short Position Collateral}}{\text{Market Value of Short Positions}}$$

Minimum initial and maintenance margin requirements apply to short positions in a similar manner to long positions.

If the investor has both long and short positions and the short account is "marked to the market,"[3] then the equity status of the entire account is given by

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[1]Nothing is really being borrowed in the short sale case. The investor is simply putting up less than 100% collateral.

[2]This is equivalent to saying that the short account has been "marked to the market."

[3]See previous footnote.

$$\text{Margin} = \frac{\begin{matrix}\text{Market} & & \text{Net} & & \text{Market Value} \\ \text{Value of} & & \text{Debit} & & \text{of Short} \\ \text{Long Collateral} & - & \text{Balance} & + & \text{Sale Collateral}\end{matrix}}{\begin{matrix}\text{Market Value} & & \text{Market Value} \\ \text{Long Positions} & + & \text{Short Positions}\end{matrix}}$$

Lower bounds on the equity status and hence upper bounds on the amount of brokers' loans than can be obtained result from the interaction of the margin accounting procedures and the minimum initial and maintenance margin requirement proscribed for various classes of securities.[1]

For expositional purposes, the maximum credit available on an existing portfolio, $\underline{X}(0)$, will first be discussed, followed by an analysis of the changes in the credit available which occur when transactions take place.

a. <u>Maximum Credit on Existing Portfolio</u>

(i) Maximum line of credit constraint

$$M(0) \leq \sum_{j=1}^{2N} (1-\beta_j^{I})X_j(0)P_j(M) \tag{6}$$

where

M(0) = Amount of margin credit outstanding before portfolio revision.

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[1] A detailed discussion of the complexities involved in margin accounting is beyond the scope of this paper. Only essential points will be treated. The interested reader is referred to the following sources,

1. <u>Federal Reserve Bulletin</u>, June 1968, "Margin Account Credit," pp. 470-481 (particularly the Appendix).

2. Thorpe and Kassouf , <u>Beat the Market</u>, Chapter 11 ("Deciphering Your Monthly Statement"), pp. 168-180.

$\beta_j^{\,I}$ = the initial margin requirement for purchases of security j, j=1, . . ., 2N.[1]

$P_j(M)$ = The collateral value of security j.[2]

When $M(0) \geq \sum_{j=1}^{2N} (1-\beta_j^{\,I})X_j(0)P_j(0)$

the account is said to be "restricted," a condition which has implications when net selling takes place.

(ii) Maintenance margin requirement

$$M(0) \leq \sum_{j=1}^{2N} (1-\beta_j^{\,M})X_j(0)P_j(0) \qquad (7)$$

where $\beta_j^{\,M}$ = the maintenance margin requirement for security j.[3]

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[1]As of December 1969, the initial margin requirements for the major classes of securities are as follows.

Class 1: Secueities listed on National Exchanges $\beta^{I}(1) = 0.80$.
Class 2: U.S. Government Securities $\beta^{I}(2) = 0.05$.
Class 3: Over the counter securities can be purchased on a cash basis only, thus $\beta^{I}(3) = 1.00$.

[2]The collateral value of a stock in an account increases in value as its market price increases. However, the reverse is not generally true. Any retreat in prices of stocks in the account will not decrease the borrowing potential until the maintenance margin condition is invoked, at which time any excess line of credit over loan value of the collateral at current prices disappears. Generally, $P(M) \geq P_j(0)$.

[3]The minimum maintenance requirements for listed stocks are:

Long Positions $\beta_L^{\,M}(1) = 0.25$

Short Positions $\beta_S^{\,M}(1) = 0.30$

(b) Purchases and Sales

When portfolio transactions take place, the affect on the maximum credit available from the broker is determined by the net purchase or sale position in each margin class (listed securities, U.S. Government securities, etc.). For net purchases in any class, the minimum initial margin must be deposited. For net sales, the reduction in the maximum line of credit depends on whether the account, net of these sales, could potentially be restricted if the full credit available was used. As long as the account could not be restricted, the maximum line of credit declines by the loan value of the collateral at current prices. Conversely, however, as long as the account could be restricted (the usual case), the maximum line of credit declines at an accelerated rate until the account could no longer be restricted. The accelerated rate, referred to as the "retention requirement," is determined by the Federal Reserve Board.[1]

(c) Maximum Credit on Revised Portfolio

(i) Net Purchase or Sale Definitional Equations

$$\sum_{[j \in \text{Class } k]} [X_j - X_j(0)]P_j(0) = NP^k - NS_1^k - NS_2^k \qquad (8)$$

$$k=1, \ldots, \bar{k}$$

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[1]Examples of retention requirements are:

1.	Listed Stocks	R(1)	=	0.70
2.	U.S.Govt. Securities	R(2)	=	0.95
3.	Over-the-counter sec.	R(3)	=	1.00

where NP^k = Net purchases in margin class k, k=1, . . ., $\bar{k}$

NS_1^k = Net sales in class k when the possibility of the account being "restricted" exists.

NS_2^k = Net sales in class k when no possibility of restriction exists.

$(NP^k)(NS_1^k) = 0$

$(NP^k)(NS_2^k) = 0$

(ii) Upper bounds constraint for the NS_1, k = 1, . . ., $\bar{k}$.

$$\sum_{k=1}^{\bar{K}} [R(k) - 1 + \beta^I(k)]NS_1^k + W$$

$$= \sum_{j=1}^{2N} (1-\beta_j^I)(P_j(M) - P_j(0))X_j(0) \qquad (9)$$

where $R(k)$ = retention requirement for margin class k

$(1-\beta^I(k))$ = loan value of collateral in margin class k

$[R(k)-1+\beta^I(k)]$ = the reduction in excess borrowing potential per dollar of net sales

right hand side = current borrowing potential in excess of loan value of the collateral at current prices.

W = Slack variable

$(W)(NS_2^k) = 0 \qquad k = 1, . . ., \bar{k}$ [1]

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[1] This condition requires that all excess borrowing potential has been eliminated before NS_2^k can be greater than zero.

(iii) Maximum line of credit on revised portfolio

$$M(1) \leq \sum_{k=1}^{\bar{k}} (1-\beta^{I}(k))NP^{k} + \sum_{j=1}^{2N} (1-\beta_{j}^{I})X_{j}(0)P_{j}(M)$$

$$- \sum_{k=1}^{\bar{k}} [R(k)NS_{1}^{k} + (1-\beta^{I}(k))NS_{2}^{k}] \qquad (10)$$

(iv) Maintenance Margin Requirement

$$M(1) \leq \sum_{k=1}^{\bar{k}} (1-\beta^{I}(k))NP^{k} + \sum_{j=1}^{2N} (1-\beta_{j}^{M})X_{j}(0)P_{j}(0)$$

$$- \sum_{k=1}^{\bar{k}} (1-\beta^{M}(k))[NS_{1}^{k} + NS_{2}^{k}] \qquad (11)$$

A term equal to M(1) - M(0) must be included in the budget equation to represent the source or use of portfolio funds resulting from the change in brokers' loans outstanding resulting from portfolio revision.

Portfolio Debt -- Unsecured Loans

The investor may be able to obtain additional funds for portfolio investment via unsecured liabilities. An example would be unsecured bank loans. These additional liabilities would be secured only by the general assets of the investor's portfolio and would depend upon his solvency at the end of the investment period for repayment. The amount of funds available from this source, as well as the amount of margin loans he can obtain, will be related to his creditors' estimates of his ability to

repay.[1]

Let $B(1)$ = the amount of unsecured loans to be held during the investment period

$B(0)$ = the original amount of unsecured debt held (before portfolio revision).

The investor's budget equation will thus contain a term $B(1) - B(0)$ to account for the funds flows resulting from changes in the unsecured debt level when the portfolio is revised.

The investor's balance sheet after portfolio revision is shown in Exhibit 2.

It is assumed that the secured margin loan $M(1)$ and the unsecured bank loan $B(1)$ are held for the duration of the investment period. The investor's creditors are assumed to limit the amount of credit offered such that the probability of the investor's terminal net worth being less than zero is virtually zero. Thus, the investor, with probability close to one, will have sufficient cash and unencumbered securities to fully meet his portfolio liabilities.

The net worth of the portfolio at the end of the investment period, $\widetilde{NW}$, is given below.

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[1]For some investors, such registered investment companies, the limits on the amount of portfolio liabilities that can be held at any time are much more explicit. The Investment Companies Act of 1940, for example, limits the liabilities of mutual funds to one half of the net asset value (net worth) of the portfolio.

Exhibit 2

BALANCE SHEET AT BEGINNING OF INVESTMENT PERIOD

(After Portfolio Revision)

Assets	Liabilities
Portfolio Market Value $\sum_{j=1}^{N} X_j P_j(0)$	Short Position $\sum_{j=N+1}^{2N} X_j P_j(0)$
	Secured Margin Loans M(1)
	Unsecured Loans B(1)
Proceeds & Collateral Deposit for Short Positions C(1)	Net Worth NW(1)

$$\widetilde{NW} = (1-T_c) \sum_{j=1}^{N} (X_j - X_{N+j})\widetilde{P}_j + (1-T_I) \sum_{j=1}^{N} (X_j - X_{N+j})\widetilde{D}_j$$

$$+ T_c \left[\sum_{j=1}^{N} (X_j - x_j^+ - X_{N+j} + x_{N+j}^+)P(A) + \sum_{j=1}^{N} (x_j^+ - x_{N+j}^+)P_j(0) \right]$$

$$+ (1+\frac{r_m}{2})C(1) - (1+r_m)M(1) - (1+r_B)B(1)$$

$$\widetilde{NW} = (\underline{X}_L - \underline{X}_S)'[(1-T_c)\underline{\widetilde{P}} + (1-T_I)\underline{\widetilde{D}}] + T_c[(\underline{X}_L - \underline{X}_S - \underline{X}_L^+ + \underline{X}_S^+)'\underline{P}(A) + (\underline{X}_L^+ - \underline{X}_S^+)\underline{P}(0)]$$

$$+ (1+\frac{r_m}{2})C(1) - (1+r_m)M(1) - (1+r_B)B(1) \tag{12}$$

where r_m = after-tax cost of brokers' loans

r_B = the after-tax cost of unsecured loans

$\underline{X}_L$ = revised portfolio vector (long positions)

$\underline{X}_S$ = revised portfolio vector (short positions)

The following notation is defined to obtain a compact expression for the variance of $\widetilde{NW}$.

Let $\sigma^T_{jj} = E[(1-T_c)\widetilde{P}_j + (1-T_j)\widetilde{D}_j]^2$

$$\sigma^T_{jj'} = E[(1-T_c)\widetilde{P}_j + (1-T_I)\widetilde{D}_j][(1-T_c)\widetilde{P}_{j'} + (1-T_I)\widetilde{D}_{j'}]$$

$$\underline{\Sigma}^T = \| \sigma^T_{jj'} \|$$

where

j=1, . . ., N

j'=1, . . ., N

The variance of the investors terminal net worth is given by

$$\sigma^2(\widetilde{NW}) = (\underline{X}_L - \underline{X}_S)' \underline{\Sigma}^T (\underline{X}_L - \underline{X}_S) \tag{13}$$

It is assumed that creditors will not supply additional funds unless they believe the investor's terminal net worth will be positive with a specified high degree of certainty. This implies that the relationship which limits the amount of debt the investor can obtain from all sources is of the form

$$P(\widetilde{NW} \leq 0) \leq \epsilon \tag{14}$$

where ϵ is a value close to zero. Additional credit will be available up to the point at which the above relationship becomes binding[1] (i.e. an equality).

This probabilistic constraint imposed by the creditors on the Investor's portfolio actions can be

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[1]In order that the maximum amount of liabilities available to the investor equal that predicted by the model, the creditors would have to have similar views regarding terminal security values as the investor. If this is not the case, then more or less debt funds will actually be available, the amount depending on the creditors' views about security performance.

The amount of credit available will also depend upon the specification of ϵ , a quantity which depends upon the degree of creditor risk aversion. An extension to this model would be to relate the rates charged on brokers' and unsecured loans to the risk of default, i.e., to the probability ϵ that the investors' terminal wealth will be less than zero.

For a discussion of probabilistic constraints, see Charnes, A. and Cooper, W. W. "Chance Constrained Programming," Management Science, Oct. 1959.

converted to a deterministic equivalent[1] under each of the assumptions made earlier about the joint distributions of terminal security values. Under assumptions B1 and B2 the distribution of terminal portfolio net worth will be normally distributed. Thus, from normal probability tables we can determine a value k such that,

$$P[\widetilde{NW} \leq E[\widetilde{NW}] + k\sigma(NW)] = \epsilon$$

Thus the condition that $P[\widetilde{NW} \leq 0] \leq \epsilon$ is equivalent to the condition that

$$E(\widetilde{NW}) + k\sigma(\widetilde{NW}) \geq 0$$

where for ϵ small k will be negative. Under assumption B3, where only the means, variances and covariances of security returns are specified, we use Tchebysheff's extended lemma[2] to obtain a deterministic equivalent of the probabilistic constraint.

By Tchebysheff's lemma

$$P\left[\frac{\widetilde{NW} - E(\widetilde{NW})}{\sigma(\widetilde{NW})} \leq k\right] \leq \frac{1}{1+k^2} \tag{15}$$

where $k < 0$.

---

[1]For a discussion of the transformation of stochastic constraints to deterministic equivalents see, Charnes, A. and Cooper W. W., "Deterministic Equivalents for Optimizing and Satisfying under Chance Constraints," Operations Research, Jan.-Feb. 1963.

[2]Harold Cramer, Mathematical Methods of Statistics, Princeton University Press, (1946), p. 256, Exercise 5.

Take

$$\epsilon = \frac{1}{1+k^2}$$

$$\therefore \quad k = -\left[\frac{1-\epsilon}{\epsilon}\right]^{1/2}$$

and it is seen that any portfolio satisfying

$$E(\widetilde{NW}) + k\,\sigma(\widetilde{NW}) \geq 0, \quad k < 0$$

will also satisfy the original probability constraint.

Thus, in each of the three cases the deterministic equivalent of the probabilistic constraint has the following form

$$(\underline{X}_L-\underline{X}_S)'[(1-T_c)\hat{\underline{P}} + (1-T_I)\hat{\underline{D}}] + T_c[(\underline{X}_L-\underline{X}_S-\underline{X}_L^{+}+\underline{X}_S^{+})\underline{P}(A) + (\underline{X}_L^{+}-\underline{X}_S^{+})\underline{P}(0)]$$

$$+ (1+\frac{r_M}{2})C(1) - (1+r_M)M(1) - (1+r_B)B(1) + k[(\underline{X}_L-\underline{X}_S)'\underline{\Sigma}^T(\underline{X}_L-\underline{X}_S)]^{1/2} \geq 0 \qquad (16)$$

$$(k < 0)$$

which is a convex function in the decision variables $X_L, X_S, X_L^{+}, X_S^{+}$.[1]

[1] With the exception of this constraint, the model developed in this paper can be specified as a quadratic programming problem. With the addition of this constraint, which is quadratic (after transferring terms and squaring both sides), the model falls into a more general class of convex programming problems. While convex programming codes exist which can handle problems for several securities, their computational efficiencies are markedly inferior to quadratic programming codes which, in reasonable amounts of time, can handle several hundred securities. In many practical cases, sufficient additional policy and legal restrictions on portfolio liabilities may exist such that this constraint will generally be non-binding. In cases where no liabilities exist, it can be ignored. (Footnote continued on next page.)

Portfolio Budget Constraint

The budget constraint insures the balancing of sources and uses of funds when the portfolio is revised. Let the portfolio cash balance be incorporated into the portfolio as security N[1]

$X_N(0)$ = initial cash balance

X_N = cash balance after portfolio revision

$F(0)$ = exogenous cash flows, which are to be optimally invested (or disbursed) when the portfolio is revised. This could include dividends accumulated from the previous investment period.

The derivation of the cash balance after revision is shown in Exhibit 3. Cash generated by selling borrowed shares (item 5) is simultaneously absorbed by increases in required deposits with the broker (item 6). Similarly, when short positions are covered, the required proceeds and collateral deposit balances are reduced, generating cash.

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[1]While X_{2N} will always be identically equal to zero, the variable is retained for convenience of notation.

(footnote continued from previous page)
Or, if we are willing to assume that the creditors' project securities prices via a market model structure,

$$\tilde{R}_{jt} = A_j + B_j\tilde{R}_{mt} + \tilde{\epsilon}_{jt}$$

where $\mathrm{Cov}\,(\tilde{\epsilon}_{jt}\,\tilde{\epsilon}_{j't}) = 0$ for all $j \neq j'$, then the standard deviation of return of a diversified portfolio can be approximated by (See William F. Sharpe, "Linear Programming Algorithm for Mutual Fund Portfolio Selection," Management Science, March 1967. pp. 501-502.)

$$\sigma\,(\tilde{R}_P) = \frac{\sum_{j=1}^{2N} X_j P_j(0) B_j}{\sum_{j=1}^{2N} X_j(0) P_j(0)}\;\sigma\,(\tilde{R}_M)$$

where $\sigma(\tilde{R}_M)$ is the standard deviation of the creditors probability distribution for $\tilde{R}_M$. This expression for $\sigma(\tilde{R}_P)$ (which is a linear function of the X_j) can be appropriately transformed to provide an estimate of $\sigma(\widetilde{NW})$ for the creditor risk constraint equation.

Exhibit 3

CALCULATION OF REVISED PORTFOLIO CASH BALANCE

No.	Component Symbol	Description
1	$+X_N(0)$	Opening Cash Balance
2	$+F(0)$	Exogenous Cash Flows
3	$-\sum_{j=1}^{2N}\left[\sum_{i=1}^{m+}\Gamma_{ji}^{+}x_{ji}^{+}+\sum_{i=1}^{m-}\Gamma_{ji}^{-}x_{ji}^{-}\right]$	Transactions' Costs
4	$-\sum_{j=1}^{N-1}(X_j-X_j(0))P_j(0)$	Security Purchases (sales)
5	$+\sum_{j=N+1}^{2N-1}(X_j-X_j(0))P_j(0)$	Short Sales (or coverage of short positions)
6	$-\ (C(1)-C(0))$	Increase in proceeds & Collateral deposits
7	$+\ (M(1)-M(0))$	Increase in Brokers' Loans
8	$+\ (B(1)-B(0))$	Increase in Unsecured Debt
9	$-\ T_C\sum_{j=1}^{N}(X_j^{-}-X_{N+j}^{-})(P_j(0)-P_j(A))$	Capital Gains Taxes

The Single Period Portfolio Selection Model --
Summary of Equations

The model for maximizing the investors expected utility of terminal net worth can now be summarized.

Select a portfolio of assets and liabilities $\underline{X}$ where

$$\underline{X} = \begin{vmatrix} \underline{X}_L \\ \underline{X}_S \\ B(1) \\ M(1) \end{vmatrix}$$

To maximize

$$Z = \theta E(\widetilde{NW}(\underline{X}) - \sigma^2(\widetilde{NW}(\underline{X})) - Y \sum_{k=1}^{\bar{k}} [(NP^k)(NS_1^k) + (NP^k)(NS_2^k) + (W)(NS_2^k)] \tag{17}$$

where $\theta \geq 0$, Y is a large positive number (e.g. 10^{10})

$$E(\widetilde{NW}(\underline{X})) = (\underline{X}_L-\underline{X}_S)'((1-T_c)\hat{\underline{P}}+(1-T_I)\hat{\underline{D}})+T_c[(X_L-\underline{X}_S-\underline{X}_L^{+}+\underline{X}_S^{+})\underline{P}(A)+(\underline{X}_L^{+}-\underline{X}_S^{+})\underline{P}(0)] + (1+\frac{r_m}{2})C(1)-(1+r_m)M(1)-(1+r_B)B(1) \tag{18}$$

$$\sigma^2(NW(\underline{X})) = (\underline{X}_L-\underline{X}_S)'\underline{\Sigma}^T(\underline{X}_L-\underline{X}_S) \tag{19}$$

Subject to

1. Budget Equation

$$F(0) - \sum_{j=1}^{2N} \left[\sum_{i=1}^{m+} r_{ji}^{+} x_{ji}^{+} + \sum_{i=1}^{m-} r_{ji}^{-} x_{ji}^{-}\right] - \sum_{j=1}^{2N} \left[X_j - X_{N+j} - X_j(0) + X_{N+j}(0)\right] P_j(0)$$

$$- \quad [C(1) - C(0)]$$

$$+ \quad [M(1) - M(0)]$$

$$+ \quad [B(1) - B(0)]$$

$$- \quad T_c \left[\sum_{j=1}^{N} (x_j^- - x_{N+j}^-)(P_j(0) - P_j(A)) \right] = 0 \qquad (20)$$

2. Proceeds and collateral deposit requirement

$$C(1) = 2 \sum_{j=N+1}^{2N} X_j P_j(0) \qquad (21)$$

3. Transactions Cost Curve Constraints

$$X_j - X_j(0) = \sum_{i=1}^{m+} x_{ji}^+ - \sum_{i=1}^{m-} x_{ji}^- \qquad j=1, \ldots, 2N \qquad (22)$$

$$x_{ji}^+ \leq \hat{x}_{ji}^+ \qquad i=1, \ldots, m+ \qquad (23)$$
$$j=1, \ldots, 2N$$

$$x_{ji}^- \leq \hat{x}_{ji}^- \qquad i=1, \ldots, m^-$$
$$j=1, \ldots, 2N$$

4. Margin Loans Restrictions

$$\sum_{j \in \text{class } k} (X_j - X_j(0))P_j(0) = NP^k - NS_1^k - NS_2^k$$

$$k = 1, \ldots, \bar{k} \qquad (24)$$

$$\sum_{k=1}^{\bar{k}} [R(k) - 1 + \beta^{I}(k)]NS_1^k + W = \sum_{j=1}^{2N} (1-\beta_j^I)(P_j(M) - P_j(0))X_j(0) \tag{25}$$

$$M(1) \leq \sum_{k=1}^{\bar{k}} (1-\beta^{I}(k))NP^k + \sum_{j=1}^{2N} (1-\beta^{I}(k))X_j(0)P_j(M) - \sum_{k=1}^{\bar{k}} [R(k)NS_1^k + (1-\beta^{I}(k))NS_2^k] \tag{26}$$

$$M(1) \leq \sum_{k=1}^{\bar{k}} (1-\beta^{I}(k))NP^k + \sum_{j=1}^{2N} (1-\beta_j^M)X_j(0)P_j(0) - \sum_{k=1}^{\bar{k}} (1-\beta^{M}(k))(NS_1^k + NS_2^k) \tag{27}$$

5. Portfolio Liabilities Constraint

$$E(\widetilde{NW}(\underline{X})) - K\,\sigma(\widetilde{NW}(\underline{X})) \geq 0, \quad K < 0 \tag{28}$$

6. Lower Bound Restrictions

$$X_j \geq 0 \qquad j=1, \ldots, 2N$$

$$x_{ji}^{+} \geq 0 \qquad j=1, \ldots, 2N, \qquad i=1, \ldots, m+$$

$$x_{ji}^{-} \geq 0 \qquad j=1, \ldots, 2N, \qquad i=1, \ldots, m-$$

$$M(1) \geq 0$$

$$B(1) \geq 0$$

The Efficient Frontier After Transactions' Costs

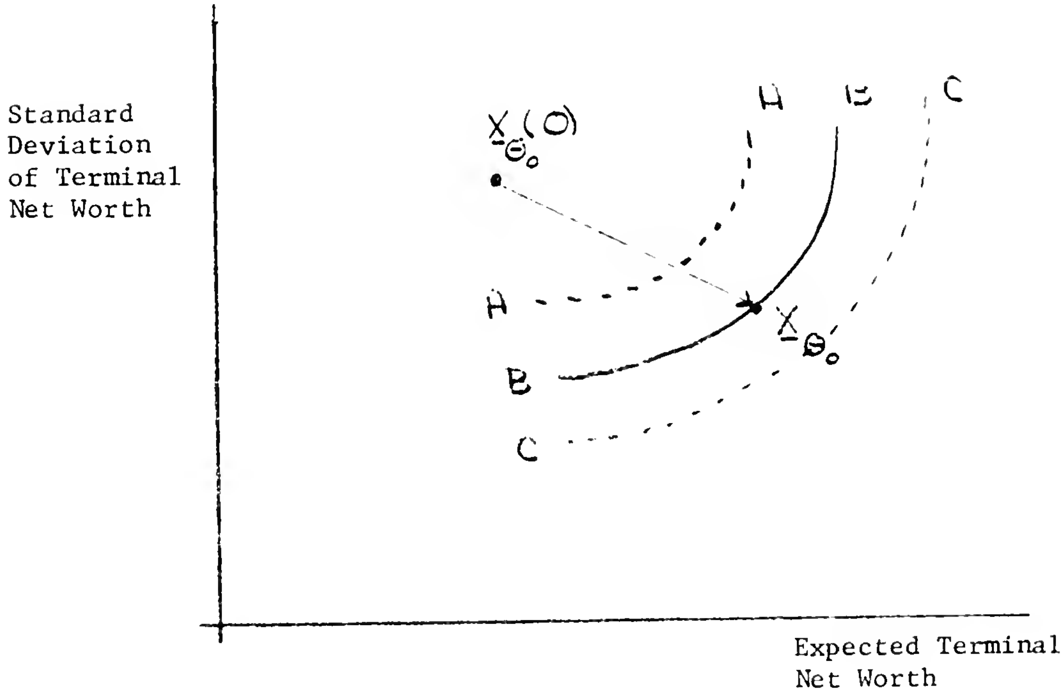
By systematically varying the parameter θ from 0 to ∞, the efficient set of portfolios for the investor's expectations about security performance can be generated. The portfolios are efficient after the transactions' costs involved in modifying the starting portfolio, $\underline{X}_{\theta_o}(0)$, to obtain a new portfolio, $\underline{X}_{\theta}$, which maximizes the investor's expected utility of terminal net worth.

The efficient frontier of portfolios which can include short sales and debt liabilities will dominate portfolios restricted from holding liabilities. A typical situation is illustrated in Figure 3, where the curve BB represents the efficient frontier including liabilities and AA represents the efficient frontier excluding liabilities. The curve CC represents the frontier BB, with the exception that transactions' costs have been ignored (assumed equal to zero). Curve CC dominates BB, but is an unrealizable alternative, due to the existence of transactions' costs for real transactions.

The shape and point of tangency with the efficient frontier of the investors indifference curve depends on the specification of the form of his utility function. If his risk preferences have not changed (i.e.,

Figure 3

THE EFFICIENT FRONTIER AFTER TRANSACTIONS' COSTS



AA - efficient frontier with no portfolio liabilities

BB - efficient frontier with portfolio liabilities

CC - efficient frontier neglecting transactions' costs

he seeks the same rate of exchange between portfolio risk and expected return, θ_o, as before) he will move from his existing portfolio $\underline{X}_{\theta_o}(0)$ to the portfolio $\underline{X}_{\theta_o}$ which is on the efficient frontier (see figure 3). If his preferences have changed, the efficient frontier contains a portfolio which is optimal for him[1] considering the costs of shifting to it from his existing portfolio.

Summary

In this paper an extended version of Markowitz's portfolio selection model has been presented. The Markowitz model has been extended to include consideration of several factors which are important in real world investment decisionmaking. These are (a) transactions' costs, including brokerage fees and volume related marketability costs; (b) short sales; (c) margin loans for security purchases and short sales; (d) unsecured portfolio debt and its relationship to the probability of insolvency to the investor.

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[1] For examples of the effects of short sales and margin loans on the two asset (plus cash) efficient frontier, see Donald D. Hester, "Efficient Portfolios with Short Sales and Margin Holdings," Chapter 3 in Risk Aversion and Portfolio Choice, Edited by Donald D. Hester and James Tobin, Cowles Foundation Monograph Number 19, John Wiley and Sons, 1967.

Appendix A

Brokerage Fees - General Case

Prior to December 5, 1968, the non-member commission rates charged by members of the New York, American and other major stock exchanges was, for a given security, directly proportional to the number of shares traded.[1] Since that time a volume discount has been introduced which applies to the portion of a transaction above 1000 shares for securities selling below \$90 per share. For securities below \$90 per share in price, the fee per "round lot" trading unit (100 shares) is less per hundred shares above 1000 shares than below. Within these respective ranges the commission charge per hundred shares remains fixed. Table 1 summarizes, on a percentage basis, commissions on 100 share transactions for securities at various prices.[2]

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[1]Fee differentials associated with odd lot trading have been ignored.

[2]To obtain the total fees associated with a transaction, state stock transfer taxes and the Securities and Exchange Commission transfer fee must be added. These fees are based on the selling price of the stock and are directly proportional to the number of shares traded, thus are easily incorporated.

Table 1

NON-MEMBER COMMISSION RATES

Price of Stock Per Share	Commission of Stock Round Lot Transactions Below 1,000 Shares	As Percentage Price Portion of Transaction Above 1,000 Shares
\$200	0.30	0.30
\$100	0.49	0.49
\$ 80	0.59	0.53
\$ 50	0.88	0.51
\$ 30	1.13	0.60
\$ 10	1.70	0.90
\$ 3	3.00	1.83

Let x_j^+ = the number of shares of security j purchased j=1, . . .,N.

x_{j1}^+ = the number of shares purchased at the higher fee rates ($x_{j1} \leq$ 1000 shares)

x_{j2}^+ = the number of shares purchased in excess of 1000 shares (x_{j1}^+ must equal 1000 before x_{j2}^+ can be greater than zero.)

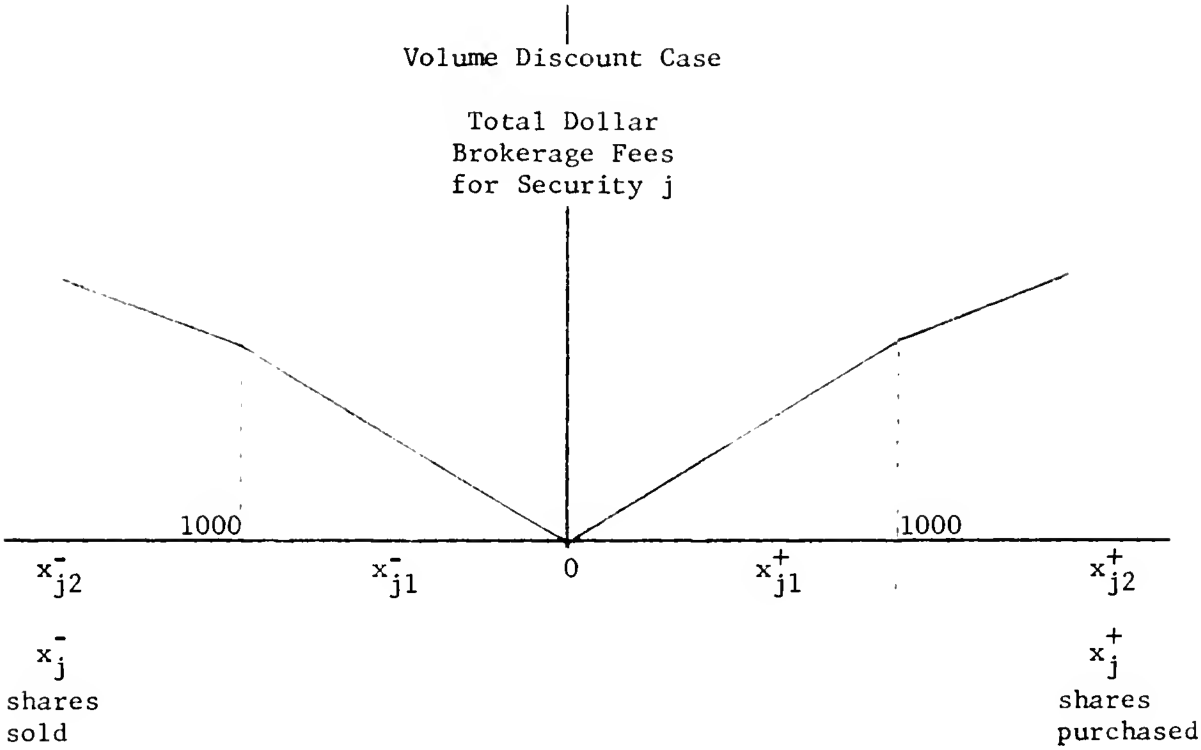
$$x_j^+ = x_{j1}^+ + x_{j2}^+$$

Similarly quantities, x_j^-, x_{j1}^- and x_{j2}^- can be defined for share sales.

The brokerage transactions cost curve illustrated in Figure 3.

Figure 3

BROKERAGE TRANSACTIONS' COST CURVE



If the above approach to defining the brokerage transactions' cost curve is to be meaningful, a means must be derived to insure that x^+_{j1} will equal 1,000 shares before x^+_{j2} takes on non-zero values. In other words, care must be taken to insure that the portfolio selection model executes the first 1,000 shares of a transaction at the higher commission rates before transacting at the lower rates which apply only to the portion of an order above 1,000 shares.

To accomplish this result, define a slack variable W_j^+ such that

$$X_{j1}^+ + W_j^+ \quad = \quad 1000 \qquad (1)$$

and

$$(X_{j2}^+)(W_j^+) \quad = \quad 0 \qquad (2)$$

Thus, whenever X_{j1}^+ is less than 1000, W_j^+ will be greater than zero ensuring X_{j2}^+ will be equal to zero as required. Only when X_{j1}^+ is identically equal to 1000 can X_{j2}^+ be greater than zero.

To obtain condition (2) within a quadratic programming framework, a term $-Y(X_{j2}^+)(W_j^+)$ is included in the objective function, where Y is an extremely large number relative to other objective function coefficients. This addition to the objective function will ensure condition (2) is maintained without the need to explicitly incorporate the non-linear constraint. A parallel analysis exists for share sales.

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